METHOD OF APPLICATION OF MOMENT DISTRIBUTION TO THE SOLUTION OF ARCHED BENTS WITH A VARYING MOMENT OF INERTIA

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By

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INTRODUCTION

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OBJECT

This work is an extension of the thesis of Cain[1] and Hansen and Jones[2] in the application of moment distribution to the solution of arched bents. This thesis will, however, deal with an arched bent of varying moment of inertia.

The fixed end moments for an arched bent whose moment of inertia varies as the secant of the angle alpha has been derived by Iselin and La Lande[3]. It will be the object of this thesis to correlate these moments and other constants as found in this thesis in order to present a solution of arched bents by moment distribution.

[1]Cain, M.G., Application of Moment Distribution to Arched Bents, Thesis, R.P.I., February, 1947

[2]B. L. Hansen and R. O. Jones, Method of Application of Moment Distribution to Solution of Arched Bents, Thesis, R.P.I., May, 1948

[3]D. G. Iselin and A. M. La Lande, A Development of an Exact Method for the Solution of Symmetrical Fixed End Parabolic Arches, Thesis, R.P.I., August, 1948

DEFINITIONS OF SYMBOLS

S or L -Length of span in feet.

h -Rise in feet - distance from springing line to crown.

H -Horizontal thrust in kips.

M_R -Fixed end moment in kips at right springing.

M_L -Fixed end moment in kips at left springing.

V -Vertical reaction.

a - $\frac{4h}{L^2}$

y -ax^2, equation of parabolic arch.

Ix -Cross sectional moment of inertia at any point along the arch.

Ic -Cross sectional moment of inertia at the center of the arch.

alpha -Angle between tangent to working line of arch, and a line parallel to springing line.

ds -Increment of length along the arch itself.

dx -Increment of length parallel to X-axis (along the springing line).

DELTA H -Total horizontal deflection of a point.

DELTA V -Total vertical deflection of a point.

THETA M -Total angular rotation of a point.

delta hv -Horizontal deflection of a point due to the application of a unit vertical force.

delta hm -The linear horizontal deflection caused by a unit moment.

delta vm -Vertical deflection of a point due to the application of a unit moment.

delta hh -The linear deflection of a point caused by a unit horizontal load at the point.

delta vv -Vertical deflection of a point due to the application of a unit vertical force.

alpha mv -Angular rotation of the member at a point due to the application of a unit vertical force.

alpha mh -Angular rotation of the member at a point due to the application of a unit horizontal force.

alpha mm - Angular rotation of the member at a point due to the application of a unit moment.

$\frac{ds}{rel.I}$ =A -A relative area of the arc segment.

x -The horizontal distance from a reference point, usually the left end of the arch.

x' -The horizontal distance from the neutral point to the point on the arch where the M/EI loading on the arch segment can be considered concentrated.

y -The vertical distance from a reference point which is usually the left springing.

y' -The vertical distance from the neutral point to the point on the arch where the M/EI loading can be considered concentrated.

$\bar{y}$ -The vertical distance from the springing line to the neutral point.

PROCEDURE

PROCEDURE

Introduction

Indeterminate rigid frames can be solved by moment distribution whether the members are of constant or varying cross section, curved or straight, provided the constants (a) stiffness at each end, (b) fixed-end moments at each end, and (c) carry-over factor at each end, are known or can be determined.

It is the purpose of this thesis to provide the constants required to solve a rigid frame whose curved top member's moment of enertia varies as the secant of the angle alpha for various rise/span ratios.

In this work a parabolic arch whose moment of inertia varies with the secant of the angle alpha and whose rise/span values are 0.04, 0.08, 0.20, 0.30, and 0.40, are dealt with. Solutions were obtained for 20 and 30 foot spans. All tabulated results are based upon a unit span of 1 foot.

The fixed-end moments as tabulated were found by using the exact solution as determined by Iselin and La Lande in their thesis entitled "A Development of an Exact Method for the Solution of Symmetrical Fixed End Parabolic Arches". In order to see of what magnitude the results which were obtained from using an approximate solution varied from the exact values, a few rise/span ratios were solved using a combination of the Conjugate Structure Method and the Neutral Point Method. The computation of the carry-over factors, absolute stiffness, and the effect of spread were made by using the General Method and Virtual Work.

The results were then combined and a sample problem was worked to illustrate the procedure in using the results obtained.

Fixed-End Moment and Thrust Determination

In the determination of the fixed-end moments, use was made of the exact solution method derived by Iselin and La Lande[1] for the solution of the parabolic arch whose moment of inertia varied as the secant of the angle alpha.

The formulae used were as follows:

$$H = \frac{15L}{8h}\left(\frac{1}{8} - v^2 + 2v^4\right)$$

$$M_R = \frac{L}{4}\left(\frac{1}{8} + v - 3v^2 - 4v^3 + 10v^4\right)$$

$$M_L = \frac{L}{4}\left(\frac{1}{8} - v - 3v^2 + 4v^3 + 10v^4\right)$$

L = Length of span

$v = \frac{w}{L}$, ratio of unit load off-center distance to length of span

w = Distance in feet from centerline to unit load

The above formulae for M_R and M_L are valid only when the unit load is placed to the left of the center or at the center. When the load is to the right of center, interchange M_R and M_L in the above equations.

The results of the above equations can be found in the section entitled "Results".

In order to determine the magnitude of error involved in using the approximate solution, and to show the method of analyzing arched bents which cannot be solved by exact mathematical formulae, the following solution is presented.

[1]See Bibliography

In the determination of the fixed-end moments, it was decided to use a combination of the Neutral Point Method and the Conjugate Structure Method, thereby eliminating the need of solving simultaneous equations.

The arch is cut at the left reaction, cantilevering out from the right end, and a weightless bracket of infinite cross section rigidly connects the cut end with the neutral point.

The arch is divided into ten equal horizontal projections. The arc length of each as solved by Hansen and Jones[1] are presented in Appendix A. It was then necessary to assume that the moment of inertia at the midpoint of the arc segments was the average moment of inertia for that segment. Computations of the moments of inertia are presented in Appendix B.

Since the elastic areas vary with the arc lengths and the average moment of inertia of the arc segments, it was necessary to compute these elastic areas. This was done by using the moment of inertia at the center of the arch, Ic, as the reference moment of inertia and computing the relative Ic/Ix for the segment. This corresponds to the width of the conjugate structure. The product of the relative Ic/Ix and the arc segment length gives the elastic area of that segment.

The Conjugate Structure Method mentioned above was developed by J. Sterling Kinney, Professor of Structural Engineering, R.P.I, Troy, New York. Its method of application is as described below.

> "The conjugate structure, for a given real structure, is identical to the real structure with regards to the lengths of the members and their relative positions. The conjugate

[1]See Bibliography

structure is positioned in a horizontal plane; and the load, which acts in a vertical direction on the conjugate structure, is the M/EI diagram of the real structure. That is, the conjugate structure is loaded with the flexural strains of the real structure.

"The end of the conjugate structure, corresponding to the end of the real structure which suffers deflection, always has a fixed support.

"The internal moment on any section of the conjugate structure is the deflection of the corresponding section of the real structure in a direction perpendicular to the lever arm used to find any particular moment.

"The shear at any section of the conjugate structure is the slope of the corresponding section of the real structure.

"Sign Convention for the Conjugate Structure

"If the flexural strain at a given section of the real structure is such as to cause tension on the outside fibers, then this flexural strain is represented as a downward load on the conjugate structure. If compression exists on the outside fibers of the real structure, the load on the corresponding section of the conjugate structure is up.

"If the moment of any section of the conjugate structure results in tension in the top fiber, the vertical deflection at the corresponding section of the real structure is down and the horizontal deflection is such as to shorten the horizontal projection of the distance between given points in the structure.

"If a section be passed through any point of the conjugate structure and if the portion of the conjugate structure to the

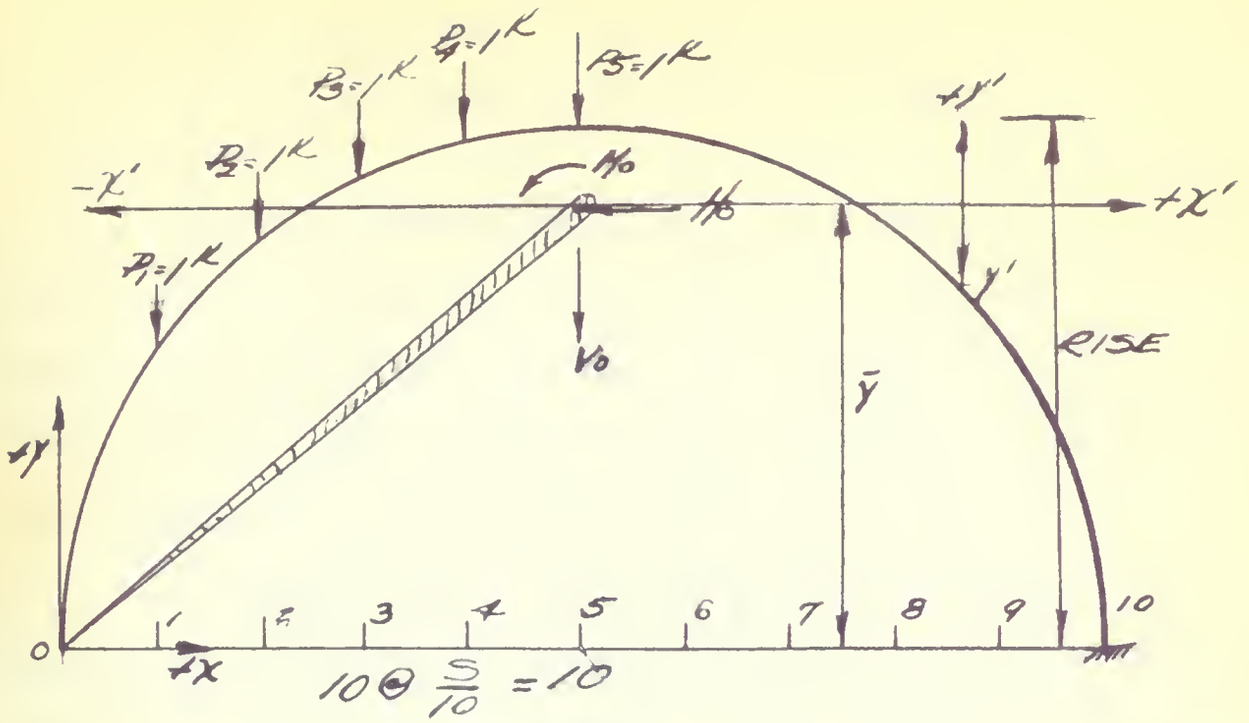
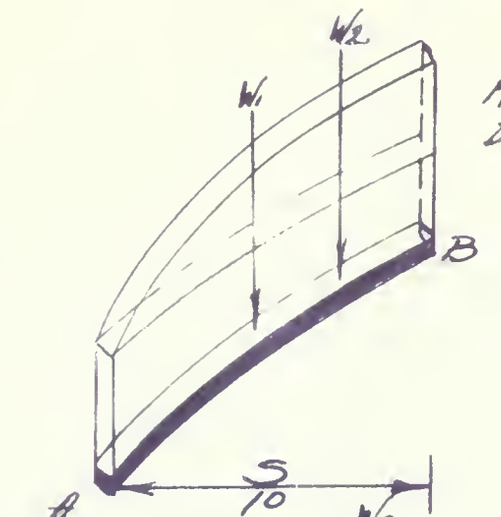
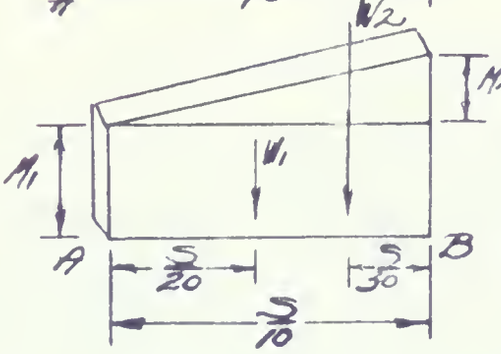


FIG. 1 PARABOLIC ARCH
REDUNDANT REACTIONS APPLIED AT NEUTRAL POINT AND VERTICAL LOADS APPLIED AT 1/10 TH SPAN POINTS.



M/EI LOADING ON AN ARC DIVIDED INTO RECTANGLE AND TRIANGLE SECTIONS



HORIZONTAL PROJECTION OF M/EI DIAGRAM LOADING ON THE ARC.
W1 - ELASTIC WT. OF RECT.
W2 - ELASTIC WT. OF TRIANGLE
M1 - MOMENT ON REAL STRUCTURE @ A
M1+M2 - MOMENT ON REAL STRUCTURE @ B

FIG 2

right of the section tends to move up, with respect to the part on the right of the section, then the rotation of the corresponding section of the real structure is clockwise." [1]

After finding the elastic areas it is necessary to compute the neutral point, the centroid of the conjugate structure. Using the centroid so found, the values, delta vv, delta hh, and alpha mm, were found by the application at the neutral point of a unit horizontal load, a unit vertical load, and a unit moment.

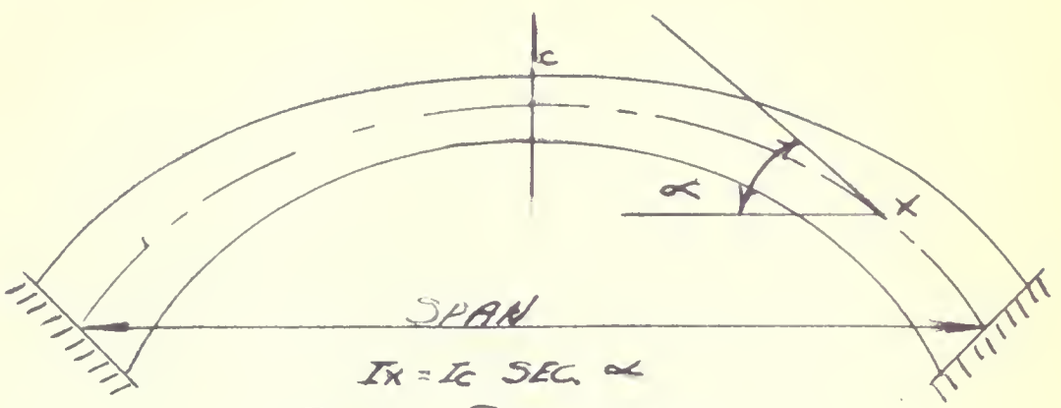
By definition of the neutral point, a horizontal force at the neutral point causes only horizontal deflection of the neutral point, a unit moment only causes rotation, and a unit vertical force causes only vertical deflection. All deflections are in the same direction as the applied unit force or moment. (See Figure 1.)

Next, unit vertical loads were placed at each of the 1/10 points across the span, and the deflections of the neutral point determined by means of the conjugate structure. These deflections were recorded as DELTA V, DELTA H, and THETA.

In order to simplify the computations for the positions of the centroids of the loaded areas, the M/EI loading diagram on a given arc segment was divided into a triangle and rectangle. The total weights of the triangle and rectangle were considered to be located on the arc and at the 1/3 point of the horizontal projection from the base of the triangle for the triangle and at the middle for the rectangle. (See Figure 2)

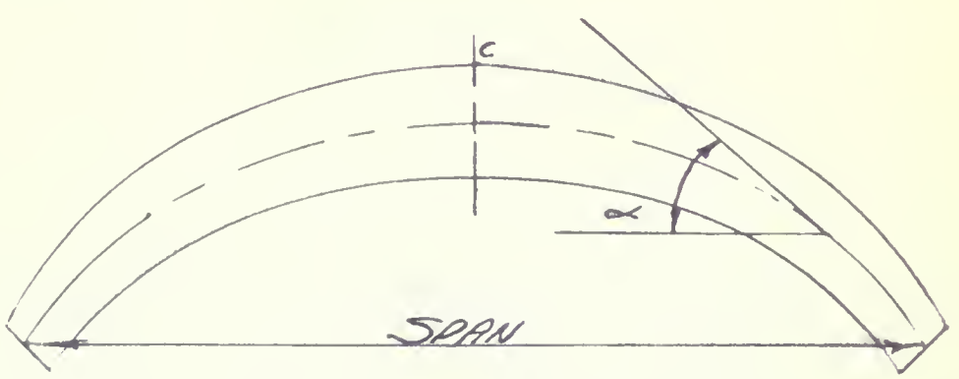
From the theories of the neutral point method, the following relationships hold true:

[1] "Indeterminate Structures" - J. S. Kinney, Sept., 1948



$I_x = I_c \sec \alpha$

REAL STRUCTURE



CONJUGATE STRUCTURE

FIGURE 3

$$Ho = -\frac{\text{DELTA H}}{\text{delta hh}} = H_L$$

$$Vo = -\frac{\text{DELTA V}}{\text{delta vv}} = V_L$$

$$Mo = -\frac{\text{THETA}}{\text{alph mm}}$$

The shear, thrust, and moment at any point on the real structure can be determined by drawing a free body diagram of the arch with the right end rigidly fixed and the left end attached to the neutral point by means of the weightless bracket, and applying the loads on the structure and the reactions at the neutral point.

Using Figure 1 and applying the reactions at the neutral point,

$$M_L = Mo + Ho\bar{y} - Vo\frac{S}{2}$$

$$M_R = Mo + Ho\bar{y} + Vo\frac{S}{2} + Px$$

with a positive moment indicating compression on the inside of the arch.

Sample computations will be found in Appendix D. It was found that the variations found by using the approximate method from the exact method were of a small magnitude.

Determination of Carry-over Factor

When an arch of varying moment of inertia is cut at the left springing, and redundant forces applied as shown in the Figure 4, deflection of the left springing will be as follows.

(1) DELTA V = $M_A$deltavm + $V_A$deltavv + $H_A$deltavh

(2) DELTA H = $M_A$deltahm + VAdeltahv + $H_A$deltahh

(3) THETA = $M_A$alphamm + $V_A$alphamv + $H_A$alphamh

Multiplying both sides of equations (1), (2), and (3) by EIc results in the following.

(4) EIcDELTA V = M_A(EIc delta vm) $+V_A$(EIc delta vv) $+H_A$(EIc delta vh)

(5) EIcDELTA H = M_A(EIc delta hm) $+V_A$(EIc delta hv) $+H_A$(EIc delta hh)

(6) EIcTHETA = M_A(EIc alpha mm) $+V_A$(EIc alpha mv) $+H_A$(EIc alpha mh)

However, to take in account the varying moment of inertia, it is necessary to use a term, "Relative $\frac{Ic}{Ix}$", to get the elastic area upon which the unit moment applied at the hinged end will act. The product of "Relative $\frac{1}{Ix}$" times ds (arc distance) will be designated as "A" which is the elastic area of the conjugate structure. (Ic is considered as unity) In this particular case, it is assumed, although it is evident that this assumption is not entirely correct, that the Ix existing at the center of the arc segment considered, is the I existing over the entire segment.

It is therefore obvious by the use of virtual work that the following equations will give us the deflections. (Σ = SIGMA)

EIc delta vm = EIc alpha mv = $\Sigma \frac{x\,ds}{I} = \Sigma\, Ax = \frac{S}{2} \Sigma\, A$

EIc delta hm = EIc alpha mh = $\Sigma \frac{y\,ds}{I} = \Sigma\, Ay = \bar{y} \Sigma\, A$

EIc alpha mm = $\Sigma \frac{ds}{I} = \Sigma\, A$

EIc delta vh = EIc delta hv = $\Sigma\, xy \frac{ds}{I} = \Sigma\, xyA = \frac{S}{2}\bar{y} \Sigma\, A$

EIc delta vv = $\Sigma\, x^2 \frac{ds}{I} + \Sigma$ Iy c.g. = $\Sigma\, Ax^2 + \Sigma$ Iy c.g.

EIc delta hh = $\Sigma y^2 \frac{ds}{I} + \Sigma$ Ix c.g. = $\Sigma\, Ay^2 + \Sigma$ Ix c.g.

It is essential that the moments of inertia of the segments be added to Ax^2 and Ay^2 values as a 4% error will occur if this is not done.

Since the carry-over factor is numerically equal to the moment induced at the fixed end of a member when a unit moment produces rotation at the hinged end, there will be no horizontal or vertical

deflections. Consequently, DELTA V and DELTA H equals 0.

Solve equations (4) and (5) simultaneously and get V_A and H_A in terms of M_A. The carry-over factor can be found by setting up a free body as indicated in Figure 5 and solving for the induced moment at the fixed end.

When the induced moment at the fixed end is such that it operates in the opposite direction from the applied moment, the carry-over factor is considered negative.

It is obvious that any moment applied at the hinged end will produce a horizontal thrust. This thrust must be known in order to apply moment distribution to the solution of the arched bent. Every time a joint is rotated and the moments are balanced out at the joint a moment will be applied to the arch. Such applied moments will change the horizontal restraining forces at the ends of the arch.

Absolute Stiffness

The absolute stiffness of a member is the value of the moment necessary, applied at the simply supported end of a member arranged as a propped cantilever, to produce a rotation of one radian at the simply supported end. This definition states that DELTA V and DELTA H must again be equal to zero. Therefore, equations(4) and (5) are again equal to zero.

Solving these equations simultaneously will give the forces induced at the hinged end in terms of the applied moment. Substituting these values in equation (6) will give EIc THETA expressed in terms of the applied moment. Set THETA equals 1 for unit rotation, and solve for M_A. This will give the absolute stiffness of the arch. After solving

several span lengths with the same rise/span ratio, it was determined that the absolute stiffness varies inversely as the span length and that the absolute stiffness remained constant for all rise/span ratios of the same span length. It is therefore concluded that the absolute stiffness is constant for any rise over span ratio and varies inversely as the span.

Spread

Any load placed on an arch causes a horizontal thrust at the ends of the arches. This thrust will cause spread in the arched bent and must be taken into consideration when solving the arched bent by moment distribution.

If a spread of DELTA H feet occurs while there is no vertical deflection or rotation of the joint, reactions will be induced at the ends of the arch. These reactions will be determined by setting equations (4) and (6) equal to zero and solving equations (4), (5), and (6) simultaneously for H_A, V_A, and M_A in terms of EIc DELTA H. After solving several span lengths for the same rise/span ratio, it was determined that V_A was equal to zero in all cases and that M_A varied inversely as the square of the span and that H_A varied inversely as the cube of the span. After solving the rise/span ratios, it was found that the induced moment varied directly as the quotient of the desired rise/span ratio over the rise/span ratio of 0.04. Integrating this knowledge, it was found that the moment varied inversely as the product of the span and the rise for that span. Similarly it was found that for the same span that the thrust varied as the square of the ratios of the rise/span ratios. Knowing this, it was found that the thrust varied inversely as the square of the rise and inversely as the span.

RESULTS

CONCLUSIONS

In this paper, tabulated data has been presented which will permit an engineer to solve any parabolic arched bent by moment distribution, provided that the moment of inertia of the arch varies as the secant of the angle alpha. Alpha is the angle between the tangent to the working line of the arch and the line at that point which is parallel to the springing line.

It was decided to omit all curves as the curves would not be large enough to permit accuracy which can be obtained by interpolation of the results.

It is believed that other arches should be solved with varying moments of inertia. This would undoubtedly save a great deal of time and energy.

As a result of this investigation, it was found that the following relationships are true.

(1) Shear is constant for all spans.

(2) Thrust varies directly as the length of span and inversely as the rise.

(3) The value of the springing moments varies directly as the length of span, regardless of the rise.

(4) The carry-over factor is constant for all spans and for all rise/span ratios.

(5) The absolute stiffness varies inversely as the span.

RESULTS BASED ON A UNIT SPAN

Fixed-end Moments

Fixed-end moments are derived from Iselin and La Lande's Formulae given on page 5 and are based on a unit load.

Load Point	M_R	M_L	V	H
1	-.01138	⅄.06058	.9718	.0304 S/h
2	-.03208	⅄.06408	.8960	.0960 S/h
3	-.04738	⅄.03678	.7840	.1656 S/h
4	-.04808	.0000	.6480	.2160 S/h
5	-.03178	-.03178	.5000	.2344 S/h

Note: Tension out is positive (⅄). The above data can be found exactly for any point from the formulae given on page 5.

Uniform Loading

Thrust in kips = $0.12\ WS^2/h$

F.E.M. in ft. kips = $0.0008\ S^2W$

W = Load/lineal foot

S = Span in feet

Absolute Stiffness = 9.068 EIc/S

Carry-over Factor = 0.3374

When joint A is free to rotate and a moment of M_A ft kips is applied at A, the following thrusts and shears are induced at A:

Thrust in kips = $0.833108\ \frac{EIc\ M_A}{h}$

Shear in kips = $0.66259\ \frac{EIc\ M_A}{S}$

When a spread, DELTA H, occurs, the following moments and thrust are developed at A:

Thrust in kips = $\frac{11.2548\ EIc\ DELTA\ H}{Sh^2}$

Moment in ft. kips = $-\frac{7.5275\ EIc\ DELTA\ H}{Sh}$

APPLICATION OF RESULTS

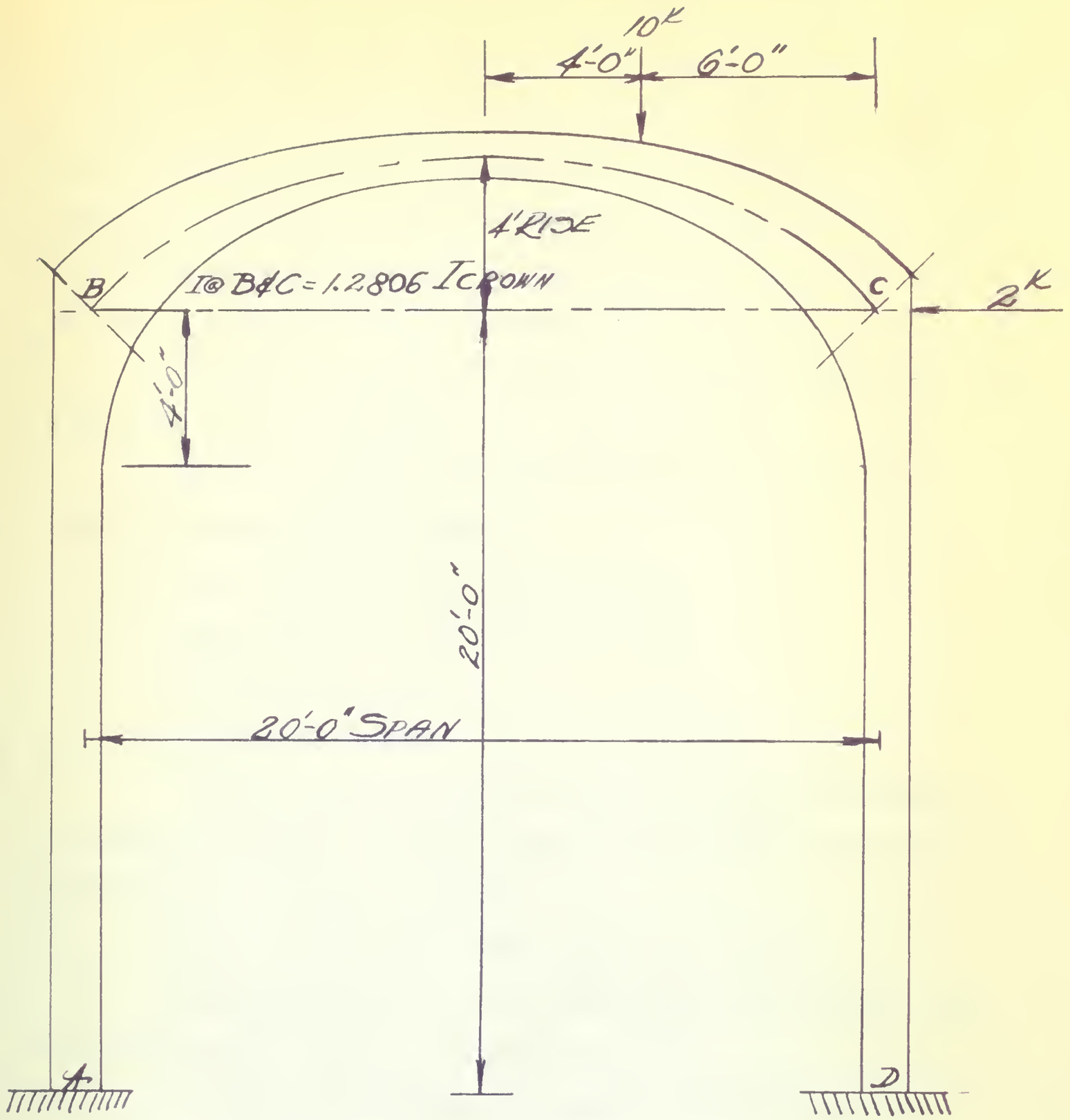


FIGURE 6

APPLICATION OF RESULTS TO THE SOLUTION OF AN ARCHED BENT WITH VARYING MOMENT OF INERTIA BY MOMENT DISTRIBUTION

Given: An arched bent whose moment of inertia at any point is equal to $Ic \frac{ds}{dx}$ has a span of 20' 0" and a parabolic rise of 4' 0". The vertical legs are unsymmetric members with a parabolic haunch at the upper end which is 4' 0" long. The moment of inertia of the upper end of the haunch is equal to the moment of inertia at the end of the arch. The lower 16'0" is equal to the moment of inertia at the center of the arch. (See Fig. 6.)

Determine the reactions at the supports and the moments at B and C which are caused by the given loads on the bent.

From the "Results", the following values for the fixed-end arched conditions can be determined.

$M_{BC} = -.0473 \times 20 \times 10 = -9.46$ ft. kips

$M_{CB} = +.0367 \times 20 \times 10 = +7.34$ ft. kips

$H_B = 8.3$ kips Right

$H_C = 8.3$ kips Left

A positive moment indicates tension on the top of the member. In moment distribution, a positive moment is that moment which tends to rotate the joint clockwise. Therefore:

$M_{BC} = -9.46$ ft. kips and $M_{CB} = -7.34$ ft. kips

The absolute stiffness of the arch is equal to .4534 EIc. The carry-over factor is -.337. (As noted on Pg. 14.)

The absolute stiffness of the parabolic haunched column is found from curves published in the "Theory and Practice of Reinforced Concrete" by C. W. Dunham.

	A	B
Absolute stiffness	.201EIc	.211EIc
Carry-over factor	.523	.498

Assume rotation of joints only and solve for the moments at all joints by moment distribution.

Joint	A	B			C		D
Member	AB	BA	BC		CB	CD	DC
K	.201	.211	.453		.453	.211	.201
K/ΣK	--	.318	.682	Balancing	.682	.318	--
C.O.F.	.523	.498	-.337	Moment	-.337	.498	.523
F.E.M.			-9.46		-7.34		
	≠1.55	≠3.01	≠6.45	≠6.45	-2.17		
			-2.18		≠6.48	≠6.48 ≠3.03	1.51
	≠0.35	≠0.69	≠1.49	≠1.49	-0.50		
			-0.12		≠0.34	≠0.34 ≠0.16	0.08
	≠0.02	≠0.04	≠0.08	≠0.08	-0.03		
					≠0.02	≠0.02 ≠0.01	
M_R	≠1.92	≠3.74	-3.74		-3.20	≠3.20	≠1.59
Bal. Mom.				≠8.02 ≠6.84			

A record of the balancing moment must be kept as this balancing moment changes the horizontal restraining force at the ends of the curved member.

For arched bents, the required horizontal restraining forces at the ends of the arch are made up of the following components:

(1) The force necessary to restrain the horizontal thrust at the ends of the arch or caused by spread in the arch.

(2) That force necessary to restrain the horizontal thrust at the ends of the arch caused by the balancing moments.

(3) That necessary to balance the shears in the column members.

(4) That force necessary to prevent sidesway when a horizontal load is applied to the bent.

A positive balancing moment at the left springing causes equal horizontal restraining forces at the two ends of the arch acting in a direction away from each other. A positive balancing moment at the right

springing causes equal horizontal restraining forces at the ends of the arch acting in the directions toward each other.

For the bent under consideration the horizontal restraining forces at the ends of the arch caused by the balancing moments is equal to 0.833108 $\frac{EI_cM_A}{n}$ which equals 0.20828 M_A.

The artificial joint restraints (AJR) required to prevent the original bent from deflecting due to sidesway or spread are equal to the horizontal restraining forces at the ends of the arch.

HRF caused by:	HRF_L	HRF_R
1. Load on the arch	⊀8.30 kips	-8.30 kips
2. Balancing Moment at B (.20828)(8.02)= 1.67	-1.67 "	⊀1.67 "
3. Balancing Moment at C (.20828) (6.84)= 1.42	⊀1.42 "	-1.42 "
4. Shear in Member AB $\frac{1.92 \nearrow 3.74}{20}$	⊀0.28 "	
5. Shear in Member CD $\frac{1.59 \nearrow 3.20}{20}$		⊀0.24 "
6. 2 kip load at C		⊀2.00 "
	⊀8.33 kips	-5.81 kips

Note: To the right is positive (⊀)

Let x equal artificial joint restraint resistant against spread (AJRsp) and let y equal artificial joint restraint resisting sidesway (AJRss). Solve for x and y.

$x \nearrow y = ⊀8.33$ kips
$x - y = 5.81$ "

$2x = 14.14$ kips
$x = 7.07$ "
$y = 1.26$ "

AJRsp = 7.07 kips acting from both ends inwards

AJRss = 1.26 kips to the right

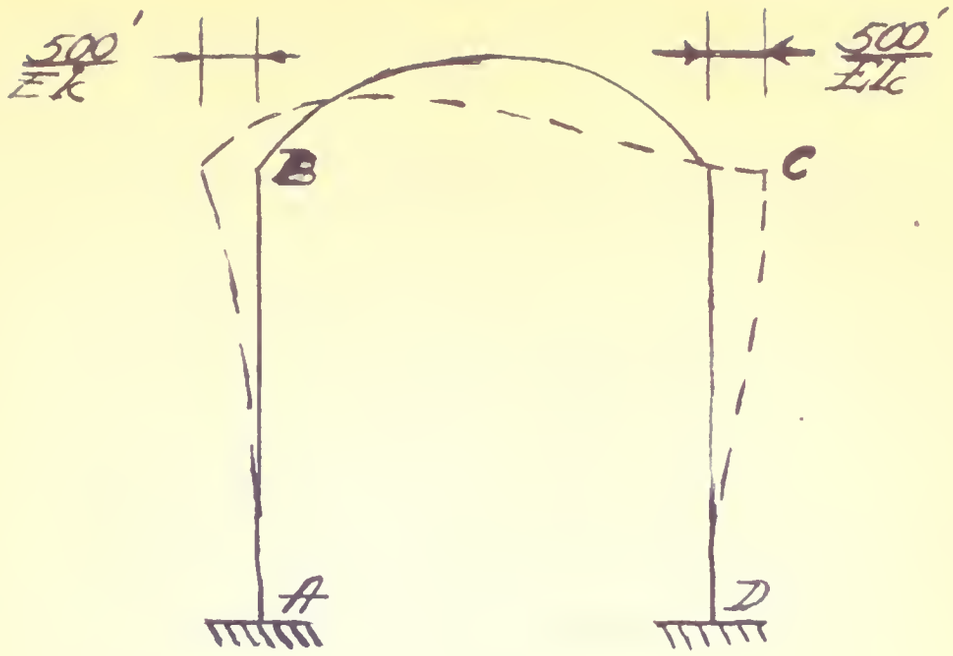


FIGURE 7

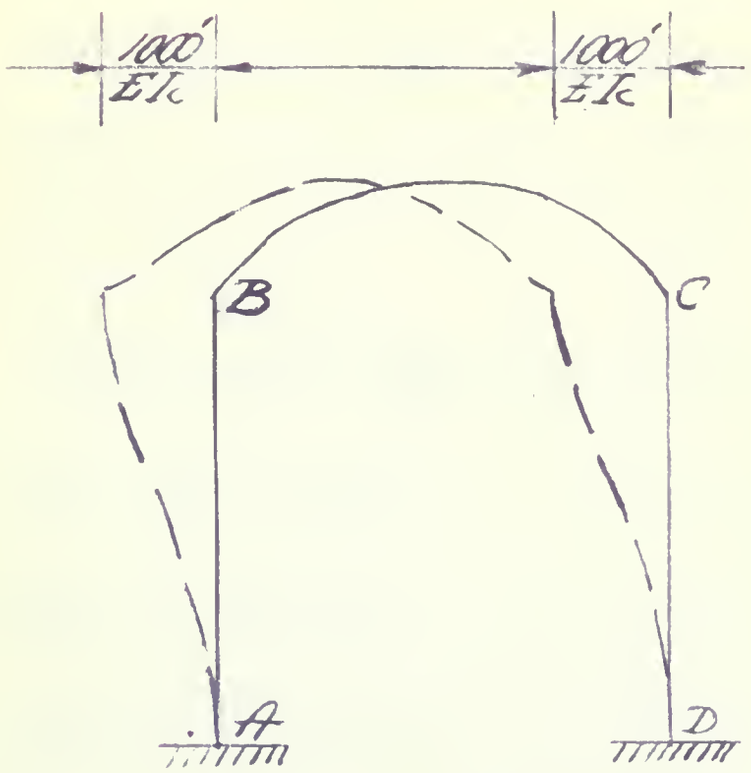


FIGURE 8

Assume a spread of 1000/EIc feet. (See Figure 7.) For a spread of the arched bent as shown, the following reactions are induced at the ends of the arch.

$$H_B = 11.2548 \frac{EIc\,DELTA\ H}{Sh^2} = \frac{11.2548\ EIc \times 1000}{20 \times 16 \times EIc} = 35.17 \text{ kips left}$$

$$H_C = 35.17 \text{ kips right}$$

$$M_{BC} = 7.5275 \frac{EIc\,DELTA\ H}{Sh} = \frac{7.5275\ EIc \times 1000}{20 \times 4 \times EIc} = +94.09 \text{ ft. kips}$$

$$M_{CB} = -94.09 \text{ ft. kips}$$

A positive moment indicates tension in the top of the arch. For the moment distribution sign convention

$M_{BC} = +94.09$ ft. kips $\qquad$ $M_{CB} = -94.09$ ft. kips

The moments induced in the ends, B and C of the column members, are equal to $K_B (1 + C_B) \frac{DELTA\ H}{L}$

$$.211\ EIc \frac{(1 + .498)}{20} \left(\frac{500}{EIc}\right) = 7.91 \text{ ft. kips}$$

where K_B = absolute stiffness at B and C_B = carry-over factor at B.

The moments induced at end A and D of the column members are equal to $K_A (1 + C_A) \frac{DELTA\ H}{L}$

$$.201\ EIc \left(\frac{1 + .523}{20}\right) \left(\frac{500}{EIc}\right) = 7.67 \text{ ft. kips}$$

$$M_{AB} = -7.67 \text{ ft. kips}$$

$$M_{BA} = -7.91 \text{ ft. kips}$$

$$M_{CD} = +7.91 \text{ ft. kips}$$

$$M_{DC} = +7.91 \text{ ft. kips}$$

Solve for the moments at the joints caused by the assumed spread.

Joint	A	B			C		D
Member	AB	BA	BC		CB	CD	DC
K	.201	.211	.453		.453	.211	.201
K/ΣK	--	.318	.682	Balancing	.682	.318	--
C.O.F.	.523	.498	-.337	Moment	-.337	.498	.523
F.E.M.	-7.67	-7.91	+94.09		-94.09	+7.91	+7.67
	-13.65	-27.41	-58.77	-58.77	+19.81		
			-15.25	+45.26	+45.26	+21.11	+10.52
	+2.42	+4.85	+10.40	+10.40	-3.50		
			-0.81	+ 2.39	+2.39	+1.11	+0.55
	+0.13	+0.26	+0.55	+0.55	-0.19		
			-0.04	+0.13	+0.13	+0.06	+0.03
		+0.01	+0.03	+0.03	-0.01		
				+0.01	+0.01		
Msp	-18.77	-30.20	+30.20		-30.19	+30.19	+18.77
Bal.Moment				-47.79 +47.79			

The consistant joint force (CJF) required at B and C to maintain the assumed spread is equal to the horizontal restraining force (HRF) at the ends of the arc.

HRF caused by:	HRF_L	HRF_R
1. Spread in the arch	-35.17 kips	+35.17 kips
2. Balancing Moment at B (.20829)(47.79)= 9.94	+9.94 "	-9.94 "
3. Balancing Moment at C (.20829)(47.79)= 9.94	+9.94 "	-9.94 "
4. Shear in Member AB $\frac{18.77 + 30.20}{20}$	-2.45 "	
5. Shear in Member CD		+2.45
CJFsp	-17.74 "	+17.74 "

Note: To the right is positive (+)

It is thus seen that a CJFsp of 17.73 kips at each end of the arch acting away from each other will produce the given result and since an AJRsp of 7.07 kips was required to restrain the original structure from spreading, the actual spread will cause the moments at the various joints to change by an amount equal to

$$\frac{AJRsp}{CJFsp} \times Msp = \frac{7.07}{17.74} \times Msp = 0.398\ Msp$$

Next assume a sideway of 1000/EIc feet to the left. (See Figure 8).

The moments induced in the ends of the column members equals:

$$K\ (1 \neq C)\ \frac{DELTA\ H}{L}$$

$M_{AB} = -15.34$ ft. kips

$M_{BA} = -15.82$ ft. kips

$M_{CD} = -15.82$ ft. kips

$M_{DC} = -15.34$ ft. kips

Solve for moments at the joints caused by sideway.

Joint	A	B				C		D
Member	AB	BA	BC			CB	CD	DC
K	.201	.211	.453			.453	.211	.201
K/ΣK	--	.318	.682	Balancing		.682	.318	--
C.O.F.	.523	.498	-.337		Moment	-.337	.498	.523
F.E.M.	-15.34	-15.82					-15.82	-15.34
	≠2.51	≠5.03	≠10.79	≠10.79		-3.60		
			-4.47		≠13.24	≠13.24	≠6.18	≠3.08
	≠0.71	≠1.42	≠3.05	≠3.05		-1.03		
			-0.24		≠0.70	≠0.70	≠0.33	≠0.15
	≠0.04	≠0.08	≠0.16	≠0.16		-0.05		
					≠0.04	≠0.04	≠0.01	
Mss	-12.09	-9.29	≠9.29			≠9.30	-9.30	-12.11
Bal. Mom.				≠14.00	≠13.98			

The consistant joint forces, CJF, required at B and C to maintain the given sideway are equal to the horizontal restraining forces at C and B and are equal to the following:

HRF caused by:	HRF_L	HRF_R
1. Balancing moment at joint B (.20829)(14.00) = 2.92	-2.92 kips	⁄2.92 kips
2. Balancing moment at joint C	⁄2.92 "	-2.92 "
3. Shear in member AB $\frac{12.09 \not 9.29}{20} = 1.07$	-1.07 "	
4. Shear in member CD	-1.07 "	-1.07 "
CJFss	-1.07 kips	-1.07 kips

Note: To the right is positive (⁄).

Thus a consistent joint force of 1.07 kips at each end of the arch acting towards the left will produce the given sidesway; and since an AJRss of 1.26 kips acting towards the right was required to keep the original structure from swaying, the actual sidesway will cause the moments at various points to change by an amount equal to:

$$\frac{AJRss}{CJFss} \times Mss = \frac{1.26}{1.07} \times Mss = 1.177\ Mss$$

Factor	M_{AB}	M_{BA}	M_{BC}	M_{CB}	M_{CD}	M_{DC}
M_R	⁄1.92	⁄ 3.74	- 3.74	- 3.20	⁄ 3.20	⁄ 1.59
0.398 Msp	- 7.47	-12.00	⁄12.00	-12.00	⁄12.00	⁄7.47
1.177 Mss	-14.25	-10.95	⁄10.95	⁄10.95	-10.95	-14.25
Final Mom.	-19.80	-19.21	⁄19.21	- 4.25	⁄ 4.25	- 5.19
Check by Conj.Struct.	-19.70	-19.40	⁄19.40	- 4.40	⁄ 4.40	- 5.20

$$H_A = \frac{19.80 \not 19.21}{20} = \frac{39.01}{20} = 1.95 \text{ kips right}$$

$$H_D = \frac{4.25 \not 5.19}{20} = \frac{9.44}{20} = .04726 \text{ kips right}$$

$$V_A = \frac{60 \not 14.96}{20} = \frac{74.96}{20} = 3.748 \text{ kips up}$$

$$V_D = \frac{140 - 14.96}{20} = \frac{125.04}{20} = 6.252 \text{ kips up}$$

APPENDIX A

TABULATION OF ARC LENGTHS

Tabulation of Arc Lengths

	Segment	20' Span	30' Span
Rise/span 0.04			
	0-1	2.02066	3.03099
	1-2	2.01262	3.01894
	2-3	2.00648	3.00972
	3-4	2.00240	3.00358
	4-5	2.00034	3.00051
Rise/span 0.08			
	0-1	2.08112	3.12169
	1-2	2.05037	3.07556
	2-3	2.02607	3.03911
	3-4	2.00927	3.01385
	4-5	2.00133	3.00205
Rise/span 0.12			
	0-1	2.17847	3.26770
	1-2	2.11035	3.16553
	2-3	2.05747	3.08621
	3-4	2.02137	3.03206
	4-5	2.00307	3.00460
Rise/span 0.20			
	0-1	2.46094	3.69140
	1-2	2.29227	3.43840
	2-3	2.15553	3.23329
	3-4	2.05874	3.08810
	4-5	2.00850	3.01275
Rise/span 0.30			
	0-1	2.94527	4.41791
	1-2	2.61420	3.92130
	2-3	2.33538	3.50307
	3-4	2.12966	3.19449
	4-5	2.01902	3.02853
Rise/span 0.40			
	0-1	3.50794	5.26190
	1-2	3.00545	4.50817
	2-3	2.56533	3.84800
	3-4	2.22470	3.33704
	4-5	2.03365	3.05047

APPENDIX B

MOMENT OF INERTIA Ix AND RELATIVE 1/Ix

Mathematical Derivation of Ix

(1) $Ix = Ic \text{ secant alpha} = Ic \frac{ds}{dx}$

The equation for the parabola is as follows:

$y = ax^2$ where $a = 4\ h/L^2$

From trigometry, it is seen that for infinitesimal lengths the following is true:

(2) $ds = (dy^2 + dx^2)^{.5}$

Therefore, we can replace dy^2 by taking the first derivative of the equation for the parabola and squaring the term.

$y = ax^2$ $\qquad dy = 2ax\ dx$

(3) $dy^2 = 4a^2\ x^2\ dx^2$

Substituting for dy^2 in formula (2) we get,

$ds = (4a^2\ x^2\ dx^2 + dx^2)^{.5}$

$ds = dx\ (4a^2\ x^2 + 1)^{.5}$

$\frac{ds}{dx} = (4a^2\ x^2 + 1)^{.5}$

Substituting for $\frac{ds}{dx}$ in equation (1) the following results:

$Ix = Ic\ (4a^2\ x^2 + 1)^{.5}$

We are assuming Ic equal to unit, therefore, the equation becomes the following:

$Ix = (4a^2\ x^2 + 1)^{.5}$

Determination of Relative Ix and Relative 1/Ix

20' span, rise/span = .08 $\quad a = \frac{4\,(1.6)}{400} = .016$

x	x^2	a^2	$4a^2x^2$	$4a^2x^2+1$	$(\ )^{.5}$	1/I
9	81	.000256	.082944	1.08294	1.0406	.9610
7	49	.000256	.050176	1.050176	1.0247	.9759
5	25	.000256	.02560	1.02560	1.0127	.9875
3	9	.000256	.009216	1.009216	1.0046	.9954
1	1	.000256	.001024	1.001024	1.0005	.9995

20' span, rise/span = .04 $\quad a = \frac{4\,(.8)}{400} = .008$

9	81	.000064	.020736	1.020736	1.0103	.9898
7	49	.000064	.012544	1.012544	1.0062	.9938
5	25	.000064	.0064	1.0064	1.0032	.9968
3	9	.000064	.002304	1.002304	1.0012	.9988
1	1	.000064	.000256	1.000256	1.0001	.9999

30' span, rise/span = .04 $\quad a = \frac{4\,(1.2)}{900} = .00533$

13.5	182.25	.0000284	.020733	1.020733	1.0103	.9898
10.5	110.25	.0000284	.012542	1.012542	1.0062	.9938
7.5	56.25	.0000284	.006399	1.006399	1.0032	.9968
4.5	20.25	.0000284	.0023036	1.002304	1.0012	.9988
1.5	2.25	.0000284	.0002560	1.000256	1.0001	.9999

20' span, rise/span = .40 $\quad a = \frac{4h}{L^2} = \frac{4(8)}{400} = .08$

9	81	.0064	2.0736	3.0736	1.7532	.5704
7	49	.0064	1.2544	2.2544	1.5015	.6660
5	25	.0064	0.6400	1.6400	1.2806	.7809
3	9	.0064	0.2304	1.2304	1.1092	.9016
1	1	.0064	0.0256	1.0256	1.0127	.9875

20' span, rise/span = .30 $\quad a = \frac{4\,(6)}{400} = .06$

9	81	.0036	1.1664	2.1664	1.4719	.6794
7	49	.0036	0.7056	1.7056	1.3060	.7657
5	25	.0036	0.3600	1.3600	1.1662	.8575
3	9	.0036	0.1296	1.1296	1.0628	.9409
1	1	.0036	0.0144	1.0144	1.0072	.9929

20' span, rise/span .20 $\quad a = \frac{4\,(4)}{400} = .04$

9	81	.0016	.5185	1.5184	1.2322	.8116
7	49	.0016	.3136	1.3136	1.1461	.8725
5	25	.0016	.1600	1.1600	1.0770	.9285
3	9	.0016	.0576	1.0576	1.0284	.9724
1	1	.0016	.0064	1.0064	1.0032	.9968

APPENDIX C

TABULATION OF ELASTIC AREAS AND CENTROID (NEUTRAL POINT) OF ARCH

Tabulation of Elastic Areas

	Segment	Arc Length	Relative 1/I	A
Rise/span .04 Span 20'	0-1	2.0207	.9898	2.0001
	1-2	2.0126	.9938	2.0001
	2-3	2.0065	.9968	2.0001
	3-4	2.0024	.9988	2.0000
	4-5	2.0003	.9999	2.0001
	0-10			20.0008
Rise/span .04 Span 30'	0-1	3.0310	.9898	3.0001
	1-2	3.0189	.9938	3.0002
	2-3	3.0097	.9968	3.0001
	3-4	3.0036	.9988	3.0000
	4-5	3.0005	.9999	3.0002
	0-10			30.0012
Rise/span .08 Span 20'	0-1	2.0811	.9610	1.9999
	1-2	2.0504	.9759	2.0010
	2-3	2.0261	.9875	2.0008
	3-4	2.0093	.9954	2.0001
	4-5	2.0013	.9995	2.0003
	0-10			20.0042
Rise/span .20 Span 20'	0-1	2.4609	.8116	1.9973
	1-2	2.2923	.8725	2.0000
	2-3	2.1555	.9285	2.0014
	3-4	2.0587	.9724	2.0019
	4-5	2.0085	.9968	2.0021
	0-10			20.0054

	Segment	Arc Length	Relative 1/I	A
Rise/span .30 Span 20'	0-1	2.9453	.6794	2.0010
	1-2	2.6142	.7657	2.0017
	2-3	2.3354	.8575	2.0026
	3-4	2.1297	.9409	2.0038
	4-5	2.0190	.9929	2.0046
	0-10			20.0274
Rise/span .40 Span 20'	0-1	3.5079	.5704	2.0009
	1-2	3.0054	.6660	2.0016
	2-3	2.5653	.7809	2.0032
	3-4	2.2247	.9016	2.0058
	4-5	2.0336	.9875	2.0082
	0-10			20.0394

Neutral Point of Arch - $\bar{y}$

Rise/span	Span	$\bar{y}$
.04	20	0.536
.08	20	1.072
.20	20	2.681
.30	20	4.0211
.40	20	5.363
.04	30	0.804
.08	30	1.608
.20	30	4.022
.30	30	6.0316
.40	30	8.0440

APPENDIX D

SOLUTION OF ARCHIMEDEAN FOR M_L, M_R, H, and V

1 Kip Vertical at Point 1

0	1	2	3	4	5	6	7	8	9	10	11
		M/EI RECTANGLE				M/EI TRIANGLE					
Sect.	A	Ht.	y'(dH)	x'(dV)	THETA 1x2	Ht.	y'(dH)	x'(dV)	THETA ½x1x6	DELTA H 5x3 9x7	DELTA V 5x4 9x8
1-2	2.0001	-	-	-	-	2	-.0916	-6 2/3	2.0001	-0.1832	-13.3340
2-3	2.0001	2	≠.064	-5	4.0002	2	≠.0898	-4 2/3	2.0001	≠0.2560 ≠0.1796	-20.0010 -9.3337
3-4	2.0000	4	≠.192	-3	8.0000	2	≠.2071	-2 2/3	2.0000	≠1.5360 ≠0.4142	-24.0000 -5.3333
4-5	2.0001	6	≠.256	-1	12.0006	2	≠.2604	-2/3	2.0001	≠3.0722 ≠0.5208	-12.0006 -1.3334
5-6	2.0001	8	≠.256	≠1	16.0008	2	≠.2498	≠1 1/3	2.0001	≠4.0962 ≠0.4996	≠16.0008 ≠2.6667
6-7	2.0000	10	≠.192	≠3	20.0000	2	≠.1751	≠3 1/3	2.0000	≠3.8400 ≠0.3502	≠60.0000 ≠6.6667
7-8	2.0001	12	≠.064	≠5	24.0012	2	≠.0364	≠5 1/3	2.0001	≠1.5361 ≠0.0728	≠120.0060 ≠10.6671
8-9	2.0001	14	-.128	≠7	28.0014	2	-.1662	≠7 1/3	2.0001	-3.5842 -0.3324	≠196.0098 ≠14.6673
9-10	2.0001	16	-.384	≠9	32.0016	2	-.4329	≠9 1/3	2.0001	-12.2886 -0.8658	≠288.0144 ≠18.6675
					144.0058				18.0007	-.8805	≠648.0303

THETA = 144.0058 ≠ 18.0007 = 162.0065

Rise - 0.8', Span - 20', Rise/span - 0.04, $\bar{y}$ - 0.536'

1 Kip Vertical at Point 2

0	1	2	3	4	5	6	7	8	9	10	11
		M/EI RECTANGLE				M/EI TRIANGLE					
Sect.	A	Ht.	y'(dH)	x'(dV)	THETA 1x2	Ht.	y'(dH)	x'(dV)	THETA ½x1x6	DELTA H 5x3 9x7	DELTA V 5x4 9x8
2-3	2.0001	-	-	-	-	2	≠.0898	-4 2/3	2.0001	≠0.1796	-9.3337
3-4	2.0000	2	≠.192	-3	4.0000	2	≠.2071	-2 2/3	2.0000	≠0.7680 ≠0.4142	-12.0000 -5.3333
4-5	2.0001	4	≠.256	-1	8.0004	2	≠.2604	-2/3	2.0001	≠2.0481 ≠.5208	-8.0004 -1.3334
5-6	2.0001	6	≠.256	≠1	12.0006	2	≠.2498	≠1 2/3	2.0001	≠3.0722 ≠.4996	≠12.0006 ≠2.6667
6-7	2.0000	8	≠.192	≠3	16.0000	2	≠.1751	≠3 1/3	2.0000	≠3.0720 ≠0.3502	≠48.0000 ≠6.6669
7-8	2.0001	10	≠.064	≠5	20.0010	2	≠.0364	≠5 1/3	2.0001	≠1.2801 ≠.0728	≠100.0050 ≠10.6671
8-9	2.0001	12	-.128	≠7	24.0012	2	-.1662	≠7 1/3	2.0001	-3.0722 -.3324	≠168.0084 ≠14.6673
9-10	2.0001	14	-.384	≠9	28.0014	2	-.4329	≠9 1/3	2.0001	-10.7525 -.8658	≠252.0126 ≠18.6675
					112.0046				16.0006	-2.7453	≠597.3613

THETA = 112.0046 ≠ 16.0006 = 128.0052

Rise = 0.8', Span = 20', Rise/span = 0.04, $\bar{y}$ = 0.536'

1 Kip Vertical at Point 3

0	1	2	3	4	5	6	7	8	9	10	11
			M/EI RECTANGLE				M/EI TRIANGLE				
Sect.	A	Ht.	y'(dH)	x'(dV)	THETA 1x2	Ht.	y'(dH)	x'(dV)	THETA ½x1x6	DELTA H 5x3 9x7	DELTA V 5x4 9x8
3-4	2.0000	-	-	-	-	2	≠.2071	-2 2/3	2.0000	≠0.4142	-5.3333
4-5	2.0001	2	≠.256	-1	4.0002	2	≠.2604	-2/3	2.0001	≠1.0240 ≠0.5208	-4.0002 -1.3335
5-6	2.0001	4	≠.256	≠1	8.0004	2	≠.2498	≠1 1/3	2.0001	≠2.0480 ≠0.4996	≠8.0004 ≠2.6668
6-7	2.0000	6	≠.192	≠3	12.0006	2	≠.1751	≠3 1/3	2.0000	≠2.3041 ≠.3502	≠36.0018 ≠6.6667
7-8	2.0001	8	≠.064	≠5	16.0008	2	≠.0364	≠5 1/3	2.0001	≠1.0241 ≠.0728	≠80.0040 10.6672
8-9	2.0001	10	-.128	≠7	20.0010	2	-.1662	≠7 1/3	2.0001	-2.5601 -.3324	≠140.0070 ≠14.6674
9-10	2.0001	12	-.384	≠9	24.0012	2	-.4329	≠9 1/3	2.0001	-9.2164 -.8658	≠216.0108 ≠18.6676
					84.0042				14.0005	-4.7169	≠522.6897

THETA = 84.0042 ≠ 14.0005 = 98.0047

Rise - 0.8', Span - 20', Rise/span - 0.04, $\bar{y}$ - 0.536'

1 Kip Vertical at Point 4

0	1	2	3	4	5	6	7	8	9	10	11
			M/EI RECTANGLE				M/EI TRIANGLE				
Sect.	A	Ht.	y'(dH)	x'(dV)	THETA 1x2	Ht.	y'(dH)	x'(dV)	THETA ½x1x6	DELTA H 5x3 9x7	DELTA V 5x4 9x8
4-5	2.0001	-	-	-	-	2	∕.2604	-2/3	2.0001	∕0.5208	-1.3335
5-6	2.0001	∕2	∕.2560	∕1	4.0002	2	∕.2498	∕1 1/3	2.0001	∕1.0240 ∕0.4996	∕4.0002 ∕2.6668
6-7	2.0000	∕4	∕1920	∕3	8.0000	2	∕.1751	∕3 1/3	2.0000	∕1.5360 ∕0.3502	∕24.0000 ∕6.6667
7-8	2.0001	∕6	∕.0640	∕5	12.0006	2	∕.0364	∕5 1/3	2.0001	∕0.7680 ∕0.0728	∕60.0030 ∕10.6672
8-9	2.0001	∕8	-.1280	∕7	16.0008	2	-.1662	∕7 1/3	2.0001	-2.0480 -0.3324	∕112.0042 ∕14.6674
9-10	2.0001	∕10	-.3840	∕9	20.0010	2	-.4329	∕9 1/3	2.0001	-7.6803 -0.8658	∕180.0090 ∕18.6676
					60.0026				12.0005	-6.1341	∕432.0142

THETA = 60.0026 ∕ 12.0005 = 72.0031

Rise - 0.8', Span - 20', Rise/span - 0.04, $\bar{y}$ - 0.536'

1 Kip Vertical at Point 5

0	1	2	3	4	5	6	7	8	9	10	11
		M/EI RECTANGLE				M/EI TRIANGLE					
Sect.	A	Ht.	y' (dH)	x'(dV)	THETA	Ht.	y'(dH)	x'(dV)	THETA	DELTA H	DELTA V
					1x2				½x1x6	5x3 9x7	5x4 9x8
5-6	2.0001	-	-	-	-	≠2	≠.2498	≠1.3333	2.0001	≠0.4996	≠2.6668
6-7	2.0000	≠2	≠.1920	≠3	4.0000	≠2	≠.1751	≠3.3333	2.0000	≠0.7680 ≠0.3502	≠12.0000 ≠6.6667
7-8	2.0001	≠4	≠.0640	≠5	8.0004	≠2	≠.0364	≠5.3333	2.0001	≠0.5120 ≠0.0728	≠40.0020 ≠10.6672
8-9	2.0001	≠6	-.1280	≠7	12.0006	≠2	-.1662	≠7.3333	2.0001	-1.5360 -0.3324	≠84.0042 ≠14.6674
9-10	2.0001	≠8	-.3840	≠9	16.0008	≠2	-.4329	≠9.3333	2.0001	-6.1442 -0.8658	≠144.0072 ≠18.6676
					40.0018				10.0004	-6.6758	≠333.3491

THETA = 40.0018 ≠ 10.0004 = 50.0022

Rise - 0.8'
Span - 20' Rise/span 0.04
$\bar{y}$ - 0.536'

1 Kip Vertical at Point 6

0	1	2	3	4	5	6	7	8	9	10	11
			M/EI RECTANGLE				M/EI TRIANGLE				
Sect.	A	Ht.	y'(dH)	x'(dV)	THETA 1x2	Ht.	y'(dH)	x'(dV)	THETA ½x1x6	DELTA H 5x3 9x7	DELTA V 5x4 9x8
6-7	2.0000	-	-	-	-	∮2	∮.1751	∮3 1/3	2.0000	∮.3502	∮6.6667
7-8	2.0001	∮2	∮.0640	∮5	4.0002	∮2	∮.0364	∮5 1/3	2.0001	∮.2560 ∮.0728	∮20.0010 ∮10.6672
8-9	2.0001	∮4	-.1280	∮7	8.0004	∮2	-.1662	∮7 1/3	2.0001	-1.0241 -.3324	∮56.0028 ∮14.6674
9-10	2.0001	∮6	-.3840	∮9	12.0006	∮2	-.4329	∮9 1/3	2.0001	-4.6082 -.8658	∮108.0054 ∮18.6676
					24.0012				8.0006	-6.1515	∮234.6781

THETA = 24.0012 ∮ 8.0006 = 32.0018

Rise - 0.8', Span - 20', Rise/span - 0.04, $\bar{y}$ - 0.536'

1 Kip Vertical at Point 7

0	1	2	3	4	5	6	7	8	9	10	11
			M/EI RECTANGLE				M/EI TRIANGLE				
Sect.	A	Ht.	y'(dH)	x'(dV)	THETA 1x2	Ht.	y'(dH)	x'(dV)	THETA ½x1x6	DELTA H 5x3 9x7	DELTA V 5x4 9x8
7-8	2.0001	-	-	-	-	≠2	≠.0364	≠5 1/3	2.0001	≠.0728	≠10.6672
8-9	2.0001	≠2	-.1280	≠7	4.0002	≠2	-.1662	≠7 1/3	2.0001	-.5120 -.3324	≠28.0014 ≠14.6674
9-10	2.0001	≠4	-.3840	≠9	8.0004	≠2	-.4329	≠9 1/3	2.0001	-3.0720 -0.8658	≠72.0036 ≠18.6676
					12.0006				6.0003	-4.7094	≠144.0072

THETA = 12.0006 ≠ 6.0003 = 18.0009

1 Kip Vertical at Point 8

8-9	2.0001	-	-	-	-	≠2	-.1662	≠7 1/3	2.0001	-0.3324	≠14.6674
9-10	2.0001	≠2	-.3840	≠9	4.0002	≠2	-.4329	≠9 1/3	2.0001	-1.5360 -0.8658	≠36.0018 ≠18.6676
					4.0002				4.0002	-2.7342	≠69.3368

THETA = 4.0002 ≠ 4.0002 = 8.0004

1 Kip Vertical at Point 9

9-10	2.0001	-	-	-	-	≠2	-.4329	≠9 1/3	2.0001 2.0001	-.8658 -.8658	≠18.6675 ≠18.6675

Rise 0.8', Span - 20', Rise/span - 0.04, $\bar{y}$ - 0.536'

1 Kip Vertical at Neutral Point

0	1	2	3	4	5	6	7	8	9	10	11
			M/EI RECTANGLE				M/EI TRIANGLE				
Sect.	A	Ht.	y'(dH)	x'(dV)	THETA 1x2	Ht.	y'(dH)	x'(dV)	THETA ½x1x6	DELTA H 5x3 9x7	DELTA V 5x4 9x8
0-1	2.0001	-8	-.3840	-9	-16.0008	-2	-.4329	-9.3333	-2.0001		≠144.0072 ≠18.6676
1-2	2.0001	-6	-.1280	-7	-12.0006	-2	-.1662	-7.3333	-2.0001		≠84.0042 ≠14.6674
2-3	2.0001	-4	≠.0648	-5	-8.0004	-2	≠.0364	-5.3333	-2.0001		≠40.0020 ≠10.6672
3-4	2.0000	-2	≠.1920	-3	-4.0000	-2	≠.1751	-3.3333	-2.0000		≠12.0000 ≠6.6667
4-5	2.0001	-	≠.2560	-1	-	-2	≠.2498	-1.3333	-2.0001		≠2.6668
5-6	2.0001	-	≠.2560	≠1	-	≠2	≠.2498	≠1.3333	≠2.0001		≠2.6668
6-7	2.0000	≠2	≠.1920	≠3	≠4.0000	≠2	≠.1751	≠3.3333	≠2.0000		≠12.0000 ≠6.6667
7-8	2.0001	≠4	≠.0648	≠5	≠8.0004	≠2	≠.0364	≠5.3333	≠2.0001		≠40.0020 ≠10.6672
8-9	2.0001	≠6	-.1280	≠7	≠12.0006	≠2	-.1662	≠7.3333	≠2.0001		≠84.0042 ≠14.6674
9-10	2.0001	≠8	-.3840	≠9	≠16.0008	≠2	-.4329	≠9.3333	≠2.0001		≠144.0072 ≠18.6676
										delta vv =	≠666.6982

Rise - 0.8', Span - 20', Rise/span - 0.04, $\bar{y}$ - 0.536'

1 Kip Horizontal at Neutral Point

0	1	2	3	4	5	6	7	8	9	10	11
		M/EI RECTANGLE				M/EI TRIANGLE					
Sect.	A	Ht.	y'(dH)	x'(dV)	THETA 1x2	Ht.	y'(dH)	x'(dV)	THETA ½x1x6	DELTA H 5x3 9x7	DELTA V 5x4 9x8
0-1	2.0001	≠.248	-.384	-9	≠.4960	≠.288	-.4329	-9.3333	≠.288	-.190464 -.124675	
1-2	2.0000	≠.024	-.128	-7	≠.048	≠.224	-.1662	-7.3333	≠.224	-.006144 -.037229	
2-3	2.0001	-	-	-5	-	≠.024 -.136	-.0147 ≠.0968	-5.903 -4.572	≠.024 -.136	-.000353 -.0131648	
3-4	2.0001	-.136	≠.192	-3	-.272	-.0960	≠.2071	-2.6667	-.096	-.052224 -.0198816	
4-5	2.0001	-.232	≠.256	-1	-.464	-.0320	≠.2611	-0.6667	-.032	-.118784 -.0083552	
										-.571275	

delta hh = 2 x -.571275 = -1.14255

Rise - 0.8', Span - 20', Rise/span - 0.04, $\bar{y}$ - 0.536'

Load at Point	Ho	Vo	Mo	$\bar{y}$ Ho	10 Vo	M_L	M_R	1 Foot Span M_L	1 Foot Span M_R
1	-.77065	.97200	-8.1000	-.41307	-9.7200	∕1.2069	-0.2264	∕0.0603	-0.0113
2	-2.40278	.89600	-6.4000	-1.28789	-8.9600	∕1.2721	-0.6426	∕0.0636	-0.0321
3	-4.12840	.78400	-4.9000	-2.21282	-7.8400	∕0.7271	-0.9493	∕0.0364	-0.0475
4	-5.36878	.64799	-3.6001	-2.8777	-6.4799	∕0.0021	-0.9657	∕0.0001	-0.0483
5	-5.8429	.50000	-2.5001	-3.1318	-5.0000	-0.6319	-0.6319	-0.0316	-0.0316
6	-5.3838	.35200	-1.6000	-2.8857	-3.5200	-0.9657	-0.0021	-0.0483	∕0.0001
7	-4.1216	.21600	-0.9001	-2.2092	-2.1600	-0.9493	∕0.7271	-0.0475	∕0.0364
8	-2.3929	.10400	-0.4000	-1.2826	-1.0400	-0.6426	∕1.2721	-0.0321	∕0.0636
9	-0.7580	.02799	-.1000	-.40629	-.2799	-.2264	∕1.2069	-0.0113	∕0.0603

$$M_L = Mo \nearrow \bar{y}\ Ho - Vo\frac{S}{2}$$

APPENDIX E

Sample Calculations for Determining:

(1) Absolute Stiffness

(2) Carry-over Factor

(3) Horizontal Thrust Due to Rotation of the Joint

(4) Effect of Spread on End Reactions

Sample calculations for determining absolute stiffness, carry-over factor, horizontal thrust due to rotation of the joint, and effect of spread on end reactions are noted in the following text.

Data: 20' Span, 6' Rise, Rise/span - .30

Elastic Areas as previously computed.

Neutral Point - 4.0211' above springing line.

See Figure 4.

For our span:

EIc delta vm = EIc alpha mv = $\frac{S}{2}$ ΣA = 10(20.0274) = 200.274

EIc delta hm = EIc alpha mh = $\bar{y}$ΣA = 4.0211(20.0274) = 80.5320

EIc alpha mm = Σ A = 20.0274

EIc delta vh = EIc delta hv = $\frac{S}{2}$ yΣA = 10(4.0211)(20.0274)=805.320

EIc delta vv = ΣAx^2 + ΣIy c.g. = 2663.2762 + 6.6756 = 2669.9518

EIc delta hh = ΣAy^2 + ΣIx c.g. = 384.6722 + 3.1720 = 387.8442

It is essential to add the moments of inertia of the individual segments to the Ax^2 and Ay^2 values. An error of 4% arises by not using this method. If the conjugate structure is used the delta vv and delta hh as found contains these values and no computations are necessary for the moments of inertia of the arc segments. This is one of the advantages of the conjugate structure method.

Computations for Ax^2, Ay^2, and Ay

Arc	A	x	x^2	Ax^2	y	y^2	Ay^2	Ay
0-1	2.0010	1	1	2.0010	1.14	1.2996	2.6005	2.2811
1-2	2.0017	3	9	18.0153	3.06	9.3636	18.7431	6.1252
2-3	2.0026	5	25	50.0650	4.50	20.2500	40.5265	9.0117
3-4	2.0038	7	49	98.1862	5.46	29.8116	59.7365	10.9407
4-5	2.0046	9	81	162.3726	5.94	35.2836	70.7295	11.9073
5-6	2.0046	11	121	242.5566	5.94	35.2836	70.7295	11.9073
6-7	2.0038	13	169	338.6422	5.46	29.8116	59.7365	10.9407
7-8	2.0026	15	225	450.5850	4.50	20.2500	40.5265	9.0117
8-9	2.0017	17	289	578.4913	3.06	9.3636	18.7431	6.1252
9-10	2.0010	19	361	722.3610	1.14	1.2996	2.6005	2.2811
0-10	20.0274			2663.2762			384.6722	80.532

20' Span, Rise/span - 0.30

Moment of Inertia of Arc Segment About Its Own Center of Gravity

Sect.	A	x	$y=ax^2$	b ya-yb	b^2	a^2	Iy c.g. $Ab^2/12$	Ix c.g. $Aa^2/12$
0-1	2.0010	8	3.84	2.16	4.6656	4	.7780	.6670
1-2	2.0017	6	2.16	1.68	2.8224	4	.4716	.6672
2-3	2.0026	4	0.96	1.20	1.4400	4	.2403	.6675
3-4	2.0038	2	0.24	0.72	0.5184	4	.0865	.6679
4-5	2.0046	0	0.00	0.24	0.0576	4	.0096	.6682
							1.5860	3.3378

Iy c.g. = 2 x 1.586 = 3.172

Ix c.g. = 2 x 3.3378 = 6.6756

Substitute in equations(4), (5), and (6), found on page 10, the following:

$EIc\Delta V = 200.2740\ M_A + 2669.9518\ V_A + 805.3218\ H_A$

$EIc\Delta H = 80.5322\ M_A + 805.3218\ V_A + 387.8442\ H_A$

$EIc\theta = 20.0274\ M_A + 200.274\ V_A + 80.532178\ H_A$

But, since DELTA V and DELTA H are equal to zero, solving equation (4) and (5) simultaneously results in $V_A = -0.033130\ M_A$ and $H_A = -0.13892\ M_A$, in which H_A is the horizontal thrust due to rotation of the joint and is equal to $-0.83552\ M_A/h$.

Using the free body diagram in the discussion and solving for M_B:

$$M_B = M_A + V_A S$$

$$M_B = M_A - 0.6626 = 0.3374$$

$$M_B = 0.3374$$

By definition the carry-over factor is equal to the moment induced at the fixed end of a member when a unit moment is applied at the hinged end. Hence, the carry-over factor = -0.3374.

The carry-over factor being negative since the induced moment at B causes tension on the same side of the member at the fixed end as the applied moment causes on the hinged end.

Substituting the values of M_A and H_A just obtained in equation (6), the following rotation at joint A occurs:

$$EIc\theta = 2.1948\ M_A$$

And when THETA is equal to one radian,

$$M_A = 0.4556\ EIc \text{ (Absolute Stiffness)}$$

$$M_A = 9.112\ \frac{EIc}{S} \text{ (Absolute Stiffness)}$$

Effect of Spread

When spread occurs with no rotation or vertical deflection at the ends of the arch, equations (4) and (6) become equal to zero. Solving them simultaneously,

$V_A = 0$

$H_A = -.248688\ M_A$

$M_A = -4.021099\ H_A$

Substituting in equation (5) and solving for H_A and M_A in terms of EIc DELTA H.

$M_A = -0.06273$ EIc DELTA H

$H_A = +.015621$ EIc DELTA H

Solving several span lengths, with the same rise/span ratio and with varied rise/span ratios, it was determined that M_A varied inversely as the product of the span and rise and that the thrust varied inversely as the product of span and the square of the rise. Therefore:

$$M_A = -7.5279\ \frac{EIc\ DELTA\ H}{Sh}$$

$$H_A = +11.2471\ \frac{EIc\ DELTA\ H}{Sh^2}$$

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Method of application of moment distribution to the solution of arched bents with a varying moment of inertia.

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